Fractional differential equations: intro, properties and new ideas

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Part I

Fractional order differential equations, as generalizations of classical order differential equations, are increasingly used in model problems in fluid flow, in finance and other areas of application, such as advection-dispersion models from hydrology. In the first part of this talk I discuss the following stationary fractional convection-diffusion model:

$$\epsilon D_{\mathcal{C}}^{\alpha} u - u' + (1+\epsilon)u = 0, \quad u(0) = 0,\ u(1) = 1,\ 1 < \alpha \leq 2,\ 0 < \epsilon \ll 1,$$

in which the fractional derivative $D_{\mathcal{C}}^{\alpha}$, replacing the traditional second derivative term, is defined 'in the sense of Caputo'.

An explicit solution of this particular model can be derived in terms of generalized Mittag-Leffler functions, which can be viewed as the fractional counterpart of the exponential function for the usual, integer order, differential equations. Interestingly, the singularly perturbed fractional boundary value problem gives rise to sharper boundary layers when the order of differentiation is decreased from two to lower values (but still bigger than one).

For the numerical approximation of the solution a straightforward finite difference method on a uniform grid is worked out and the numerical solution is compared with the exact solution. In this case a fractional mesh-Péclet number may be defined, which is of importance for the monotonicity condition of the numerical solution. Numerical experiments will illustrate some of these ideas.

Part II

In the second part, some new ideas and results on the fractional heat equation are presented. This involves, among others, fractional powers of finite-difference matrices for which a special (stable) 'matrix-Newton' procedure has to be used. Numerical results indicate a strong link with the traditional definitions of fractional derivatives, such as the Caputo, Riemann-Liouville and Grünwald-Letnikov formulations. Although the results are promising, still a lot of work has to be done to prove the different steps in the proposed method.